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## LETTER TO THE EDITOR

# Critical behaviour near multiple junctions and dirty surfaces in the two-dimensional Ising model

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**Abstract.** We consider  $m$  two-dimensional semi-infinite planes of Ising spins joined together through surface spins and study the critical behaviour near to the junction. The  $m = 0$  limit of the model—according to the replica trick—corresponds to the semi-infinite Ising model in the presence of a random surface field (RSF1). Using conformal mapping, second-order perturbation expansion around the weak- and strong-coupled planes limits and differential renormalization group, we show that the surface critical behaviour of the RSF1 model is described by Ising critical exponents with logarithmic corrections to scaling, while at multiple junctions ( $m > 2$ ) the transition is of first-order. There is a spontaneous junction magnetization at the bulk critical point.

The critical behaviour of systems near a plane where translational invariance is broken is of considerable recent interest [1, 2]. The prototype of these problems is represented by the critical phenomena at a free regular surface (semi-infinite criticality), and in more complex problems the effect of a perturbation (e.g. surface coupling enhancement, interfaces, defects, random surface fields, etc.) can be analysed by relevance-irrelevance type criteria [3-5]. Such a stability analysis, however, does not work for the two-dimensional Ising model in the case of marginal perturbations caused by a defect line [3, 4] or a random surface field (RSF1) [5], when the defect exponent  $y_d = 0$ . In the former case non-universal critical behaviour was found by an exact calculation [6, 7], while for the RSF1 model no definite answer is known yet. One of our purposes in the present letter is to clarify the critical behaviour of the RSF1 model.

To study this problem we introduce a series of models consisting of  $m$  semi-infinite planes of Ising spins where the spins at different surfaces are joined together by nearest neighbour couplings (see figure 1(a)). (We note that in a recent paper Indekeu and Nikas [8] introduced a junction as a product of surface spins and studied the wetting phenomena in the  $m = 3$  system in the frame of Landau theory.) The Hamiltonian of the system is given by

$$H = \sum_{p=1}^m H_p + V \quad (1)$$

where

$$-\beta H_p = \sum_{y=-\infty}^{\infty} \sum_{x=1}^{\infty} [J_y \sigma^p(y, x) \sigma^p(y+1, x) + J_x \sigma^p(y, x) \sigma^p(y, x+1)] \quad (2)$$

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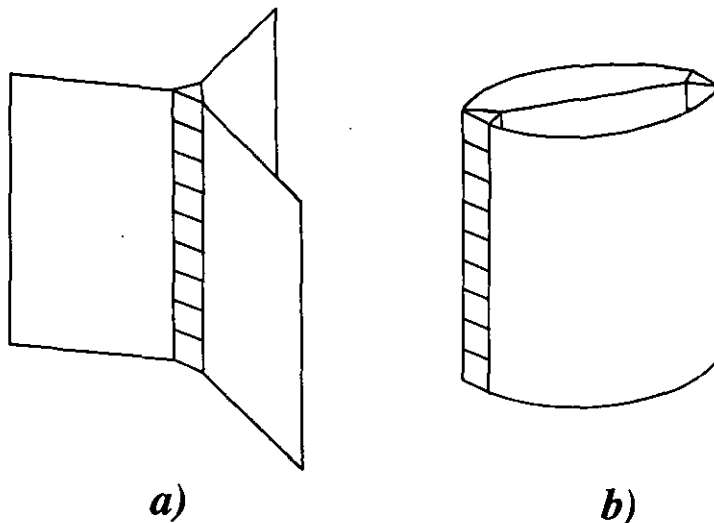


Figure 1. (a)  $m = 3$  semi-infinite planes of Ising spins joined together by surface spins and (b) the corresponding system in the strip geometry.

and

$$-\beta V = J \sum_{y=-\infty}^{\infty} \sum_{p < p'} \sigma^p(y, 1) \sigma^{p'}(y, 1). \quad (3)$$

Here  $\sigma^p(y, x) = \pm 1$  are Ising spins at position  $y, x$  on the  $p$ th plane, and  $\beta = 1/k_B T$ .

For  $m = 1$  and  $m = 2$  we obtain a semi-infinite system and two semi-infinite systems joined by a defect line, respectively, while for  $m = 3$  as a possible physical realization one can imagine interacting magnetic ions segregated along various planar grain boundaries, three of which meet along a linear junction. Due to pair interactions in the junction (equation (3)) the perturbation represented by  $V$  is *marginal* in the ordinary surface transition point for all  $m \neq 1$ ; thus to clarify the actual critical behaviour one needs detailed investigations.

Now we show that the multiple junction problem is connected to the RSFI model defined by the Hamiltonian:

$$-\beta \tilde{H}_p = -\beta H_p + \sum_{y=-\infty}^{\infty} h(y) \sigma^p(y, 1) \quad (4)$$

where the random surface field  $h(y)$  has a Gaussian distribution:

$$P[h(y)] = \frac{1}{\sqrt{2\pi\Delta^2}} \exp\left(-\frac{h^2(y)}{2\Delta^2}\right). \quad (5)$$

Since the disorder is quenched, it is the free energy rather than the partition function which must be averaged. Using the replica trick:  $\langle \log Z \rangle = \lim_{m \rightarrow 0} [(\langle Z^m \rangle - 1)/m]$  one can easily show that the effective Hamiltonian of the problem is just the  $m \rightarrow 0$  limit of equations (1)-(3) with  $J = \Delta^2$ .

In general we are interested in the critical behaviour near to the junction, e.g. we look for the critical exponent  $\eta_m$  describing the decay of spin-spin correlations  $\langle s^p(0, 1) s^p(y, 1) \rangle \propto |y|^{-\eta_m}$  when the system is at the bulk critical point. Now we suppose

that critical correlations in the model transform covariantly under a conformal transformation. It is exactly known for  $m = 1$  and  $m = 2$  [9], on the other hand the gap-exponent relation (6) might be valid, even if the system is not conformally invariant [10]. In the following we map the system onto the strip geometry [11], where the calculation is usually simpler to perform. Denoting the points on the  $p$ th plane by the complex number  $z_p$ , then the conformal transformation  $w_p = (L/\pi) \log z_p$ ,  $p = 1, 2, \dots, m$ , maps the semi-infinite planes onto strips of width  $L$ , and the surface spins at both ends of the strips are connected to each other with the same type of coupling as in the plane geometry, equation (3) (see figure 1(b)).

The critical exponents in the strip geometry can be calculated from the finite-size behaviour of the correlation length [11]. More precisely we consider the extreme-anisotropic limit [12] of the model, when the transfer matrix along the strip is expressed as  $T = \exp(-a\hat{H})$ , where  $a$  is the lattice spacing and  $\hat{H}$  is a quantum Hamiltonian. Then, following Cardy's derivation [13] one can show that the spectrum of the critical Hamiltonian operator  $\hat{H}$  in the large- $L$  limit is given as:

$$E_i - E_0 = \frac{\pi}{L} v_s x_i \tag{6}$$

where  $E_0$  and  $E_i$  are the ground state and the  $i$ th excited state of  $\hat{H}$ , respectively, and  $v_s$  is a normalizing factor, the so-called sound velocity. The set of critical dimensions  $x_i$  describes the decay of correlations of scaling operators  $\Phi^i$  along the junction in the plane geometry:  $\langle \Phi^i(z)\Phi^i(0) \rangle \propto |z|^{-2x_i}$ . For the spin operator we have  $\eta_m = 2x_s$ . The spectrum in (6) usually has a tower-like structure; the levels in the same tower differ by an integer from the lowest one:  $x_i = x_i^0 + l$ ;  $l = 0, 1, 2, \dots$ , and  $x_i^0$  is the critical dimension of a primary operator [11].

For our model the quantum Hamiltonian  $\hat{H}$  is given as

$$\hat{H} = \sum_{p=1}^m \hat{H}_p + \hat{V} \tag{7}$$

$$\hat{H}_p = - \sum_{x=1}^{L-1} \sigma_z^p(x) \sigma_z^p(x+1) - h \sum_{x=1}^L \sigma_x^p(x) \tag{7a}$$

$$\hat{V} = -\lambda_1 \sum_{p < p'} \sigma_z^p(1) \sigma_z^{p'}(1) - \lambda_L \sum_{p < p'} \sigma_z^p(L) \sigma_z^{p'}(L). \tag{7b}$$

In (7a, b)  $\sigma_x^p$  and  $\sigma_z^p$  are Pauli matrices at chain  $p$  on site  $x$ , the bulk critical point corresponds to  $h = 1$  [12], while in the units used in (7)  $v_s = 2$  [14]. Note, that we put different values of the couplings at both ends of the chains ( $\lambda_1, \lambda_L$ ): the model in (1)-(3) corresponds to  $\lambda_1 = \lambda_L = \lambda$ .

We mention that one may also consider systems composed of planes having two free surfaces, say at  $x = 0$  and  $y = 0$ . For these 'half-infinite' systems in the Hamiltonian equations (2), (3) the summations over  $y$  run from 0 to  $\infty$ . Now in the transformed geometry the strips are coupled only at one edge and the others are free; furthermore in the transformation  $\pi$  is replaced by  $\pi/2$ , the angle at the corner [11]. In this case in (7b) we have  $\lambda_1 = \lambda$  and  $\lambda_L = 0$ .

To calculate the spectrum of (7) one has to keep in mind the possible presence of strong logarithmic corrections to scaling, and therefore to try to push analytical calculations as far as possible. Since for general value of  $m$  no exact solution can be found, we performed a perturbation expansion around the uncoupled chains limit and sum up the most diverging contributions by using the differential renormalization group technique.

The actual calculation of the coefficients of the weak-junction expansion is rather cumbersome; therefore here we present only the final result of the second-order calculation; details of the derivation will be published elsewhere [15]. The first gap of the system—corresponding to magnetic excitations—in the large- $L$  limit is given by:

$$E_1 - E_0 = \frac{2\pi}{L} \left[ \frac{1}{2} - (m-1) \frac{1}{\pi} (\lambda_1 + \lambda_L) + (m-1) \frac{2}{\pi^2} \lambda_1 \lambda_L - (m-1)(m-2) \left( \frac{2}{\pi^2} \log L - C \right) (\lambda_1^2 + \lambda_L^2) + \dots \right] + O(1/L^2) \quad (8)$$

where  $C = 0.03672\dots$  is a constant, and according to (6) the quantity in the square bracket is just the leading magnetic exponent  $x_s$ . The main observation concerning equation (8) is that in first order  $x_s$  is regular and coupling dependent, but the second-order coefficient diverges as  $\log L$ . The perturbation series for higher gaps shows the same qualitative picture. On the basis of the differential renormalization group [16] one assumes that the higher-order terms of the expansion in (8) are divergent too, but these singular terms sum up to a regular contribution when the perturbation is marginally irrelevant.

To show this we start writing the differential renormalization group equations of the problem under a change of the length scale  $e^l$  in the form [16]:

$$\frac{d\lambda_1}{dl} = y_d \lambda_1 + b \lambda_1^2 + O(\lambda_1^3) \quad (9)$$

$$\frac{d\lambda_L}{dl} = y_d \lambda_L + b \lambda_L^2 + O(\lambda_L^3)$$

where the defect exponent  $y_d = 0$  and the perturbation is marginally irrelevant and marginally relevant for  $b < 0$  and  $b > 0$ , respectively. The solutions of (9) are given as  $\lambda_1(l) = \lambda_1 / (1 - b \lambda_1 l)$  and  $\lambda_L(l) = \lambda_L / (1 - b \lambda_L l)$ , respectively. From the transformation form of the inverse correlation length

$$\xi_n^{-1}(\lambda_1, \lambda_L, L^{-1}) = e^{-l} \xi_n^{-1}(\lambda_1(l), \lambda_L(l), L^{-1} e^l)$$

one gets the scaling prediction of the gaps for finite systems:

$$E_n - E_0 = L^{-1} \Phi_n \left( \frac{\lambda_1}{1 - b \lambda_1 \log L}, \frac{\lambda_L}{1 - b \lambda_L \log L} \right). \quad (10)$$

Expanding  $\Phi_n$  up to second order in powers of  $\lambda_1, \lambda_L$  one can verify that its form is indeed compatible with the expansion in (8) with  $b = (2/\pi)(m-2)$ . One can verify similarly that the scaling form (10) is also valid for higher gaps with the same value of  $b$ . This fact is due to the structure of the second-order *degenerate* perturbation calculation: different gaps are represented by different secular matrices, but all the matrix elements are the same function of  $\lambda_1, \lambda_L$  and  $L$  independently of the matrix.

Thus we arrive at the conclusion, that the relevance-irrelevance behaviour of the perturbation caused by the junction depends on the value of  $m$ : for positive defect couplings it is marginally irrelevant and marginally relevant for  $m < 2$  and  $m > 2$ , respectively. In the borderline case  $m = 2$  the perturbation is fully marginal, the critical exponents are coupling dependent [6, 7].

For  $m < 2$  in the large- $L$  limit:  $|b|\lambda_1 \log L \gg 1$ ,  $|b|\lambda_L \log L \gg 1$ , the finite-size behaviour of the magnetic exponent is given by:

$$x_s = \frac{1}{2} - \frac{m-1}{2-m} \frac{1}{\log L} + \dots \quad (11)$$

Concerning the surface transition of the RSFI model ( $m=0$ ) we get Ising critical exponents with logarithmic correction to scaling

$$x_s^R = \frac{1}{2} \left( 1 + \frac{1}{\log L} + \dots \right) \quad (11a)$$

which is one of the main results of our paper. The finite-size form of the exponent  $x_s^R$  in (11a) gives a theoretical basis to deduce effective exponents from MC simulations or from transfer matrix calculations. We note, that similar behaviour—exponents of the pure model with logarithmic corrections—has been observed for the magnetization and the susceptibility of the random bond two-dimensional Ising model [17], which can be considered as the bulk analogue of the RSFI model.

Now we turn to the problem of multiple junctions ( $m > 2$ ) with ferromagnetic interactions. In this case the perturbation is *marginally relevant*, consequently the critical behaviour is controlled by a *new fixed point*. We believe that the system for non-zero defect couplings undergoes a first-order transition, there is a spontaneous junction magnetization at the bulk critical point. To prove this conjecture we consider the strong defect limit of the problem and investigate the stability of the corresponding fixed point.

In the limit  $\lambda_1 \rightarrow \infty$  and/or  $\lambda_L \rightarrow \infty$  all the spins in the junctions are parallel, consequently the ground state of the system is two-fold degenerate, i.e., the lowest gap is zero and there is a spontaneous junction magnetization of the system. In this limit the spectrum of  $\hat{H}$  can be constructed as the direct product of  $m$  Virasoro algebras corresponding to the spectrum of the Ising model at the extraordinary transition point [11].

The stability of this fixed point is determined through the size dependence of the lowest gap for finite defect couplings. In the *half-infinite* case,  $\lambda_1 \gg 1$ ,  $\lambda_L = 0$  in first order of  $1/\lambda_1$  the gap is given by [14]

$$E_1 - E_0 = \frac{m}{(m-1)2^{m-2}} \frac{1}{\lambda_1^{m-1}} \left( \frac{1}{L} \right)^{m/2} + \dots \quad (12)$$

while for non-zero  $\lambda_L$  the leading finite-size dependence of the gap remains the same with a more complicated prefactor than in (12).

According to (12) for  $m > 2$  at finite couplings the first gap vanishes faster than  $1/L$ , thus from (6)  $x_s = 0$  and the perturbation is irrelevant. The ground state of the system is degenerate and there is a spontaneous junction magnetization at the bulk critical point in accordance with our claim. We note that a similar mechanism has been observed for other problems where surface or interface ordering take place at the bulk critical point: the lowest gap vanishes algebraically with the size of the system [18–20]. For  $m=2$  according to (12) the perturbation is *marginal*,  $x_s$  is coupling dependent, while for  $m < 2$  the gap vanishes slower than  $1/L$ , thus the extraordinary transition is unstable in this region.

Finally we turn to discuss the critical behaviour near to a junction with antiferromagnetic (AF) interactions. In this case according to equations (9) the signs of  $b$  for marginally irrelevant and marginally relevant perturbations are interchanged with

respect to the ferromagnetic junction. As a consequence for  $m > 2$  the perturbation at the ordinary surface transition point is *marginally irrelevant*; the critical exponent  $x_s$  together with the logarithmic finite size correction is given by (11).

It is interesting to study the strong junction limit for AF couplings, which gives qualitatively different results for  $m = \text{odd}$  and  $m = \text{even}$ . For  $m = \text{odd}$ , due to frustration, there is no extraordinary transition even for infinitely strong couplings in the defect: the first gap is proportional to  $L^{-1/2}$  [15]. This phenomena resembles the absence of phase transition at  $T=0$  for super-frustrated models [21]. On the other hand for  $m = \text{even}$  there is an extraordinary transition for infinitely strong AF couplings. The perturbation to the first gap is marginal in leading order:  $E_1 - E_0 \propto 1/(\lambda L)$ , thus further analysis is needed to decide on the type of transition in this case.

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## References

- [1] Binder K 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (London: Academic)
- [2] Diehl H W J 1986 *Phase Transitions and Critical Phenomena* vol 10, C Domb and J L Lebowitz (London: Academic)
- [3] Bray A J and Moore M A 1977 *J. Phys. A: Math. Gen.* **10** 1927
- [4] Burkhardt T W and Eisenriegler E 1981 *Phys. Rev. B* **24** 1236
- [5] Diehl H W and Nüsser A 1990 *Z. Phys. B* **79** 69
- [6] Bariev R 1979 *Zh. Eksp. Teor. Fiz.* **77** 1217
- [7] McCoy B M and Perk J H H 1980 *Phys. Rev. Lett.* **44** 840
- [8] Indekeu J and Nikas Y *Preprint Leuven*
- [9] Turban L 1985 *J. Phys. A: Math. Gen.* **18** L325  
Henkel M, Patkós A and Schlottmann M 1989 *Nucl. Phys. B* **314** 609
- [10] Guim I, Burkhardt T W and Xue T 1990 *Phys. Rev. B* **42** 10298
- [11] Cardy J L 1987 *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J L Lebowitz (London: Academic)
- [12] Kogut J 1979 *Rev. Mod. Phys.* **51** 659
- [13] Cardy J L 1984 *J. Phys. A: Math. Gen.* **17** L385
- [14] Burkhardt T W and Guim I 1984 *J. Phys. A: Math. Gen.* **17** L385
- [15] Iglói F, Turban L and Berche B, unpublished
- [16] Cardy J L 1986 *J. Phys. A: Math. Gen.* **19** L1093
- [17] Wang J S, Selke W, Dotsenko V I S and Andreichenko V B 1990 *Europhys. Lett.* **11** 301
- [18] Burkhardt T W and Iglói F 1990 *J. Phys. A: Math. Gen.* **23** L633
- [19] Iglói F, Berche B and Turban L 1990 *Phys. Rev. Lett.* **65** 1773
- [20] Iglói F 1990 *Phys. Rev. Lett.* **64** 3035
- [21] Sütő A 1981 *Z. Phys.* **44** 121